On the topology of the group of invertible elements

– A Survey –

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The topological structure of the group of invertible elements in a unital Banach algebra (regular group for short) has attracted topologists from the very beginning of homotopy theory. Be it for its own sake simply to show the instrumental power of newly invented methods or because there were important applications, the most notable one being perhaps the Atiyah-Singer index theorem whose topological pillar is Bott's periodicity theorem for the homotopy groups of the stable general linear group. Recently, operator K-theory, which is the homotopy theory of the stable regular group of a C^*-algebra, has been used to obtain index theorems in a more general setting. While the properties that are needed in index theory are by now quite well understood, since there one only makes use of the stable regular group, the topological structure of the regular group of a C^*-algebra itself is less well studied. Here we want to survey the present state. Although today operator K-theory is merged in KK-theory we do not include these new developments since they would take us to far and beyond our purpose.

The prototype of a regular group is the general linear group $GL(n, \mathbb{F})$ of invertible $n \times n$-matrices with entries from $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Exploring its homotopical structure is intimately linked to the development of algebraic topology, and even nowadays a full understanding is out of reach. Without exposing the highly sophisticated methods which are used to pursue this problem, we will give a short historical account in the first section. In the second section we look at the general linear group $GL(E)$ of invertible continuous linear operators on a Banach space E, and also as an intermediate step from finite to infinite dimensions at the Fredholm group $GL_c(E)$, the subgroup of $GL(E)$ that consists of perturbations of the identity by compact operators. General Banach algebras will be studied in section three, in particular, commutative Banach algebras which for nearly forty years have taken an independent development because of the close relation to complex analysis. In the last section we deal exclusively with C^*-algebras. We review the important tools and results from operator K-theory and then go on to discuss nonstable K-theory. However, we have to be selective since operator K-theory is still rapidly expanding which makes it impossible to present all of the known K-groups. For more detailed information the reader is referred to the bibliography which is kept to a minimum and contains only articles that are well-suited for cross reference so that most of the information can be retrieved from these. In each section we also consider the topology of some associated spaces such as the space of idempotents and the Grassmannian.

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1. The classical groups

The simplest Banach algebras are the finite dimensional matrix algebras $M(n,\mathbb{F})$ ($=M(n,n,\mathbb{F})$ if we denote by $M(n,m,\mathbb{F})$ the space of $n\times m$-matrices with real, complex or quaternionic entries). The regular group of $M(n,\mathbb{F})$ is the general linear group $GL(n,\mathbb{F})$. The classical groups (a term coined by Hermann Weyl) are subgroups of $GL(n,\mathbb{F})$ and defined as follows. If $^-$ denotes conjugation in $\mathbb{F}$ and ^{t} transposition in $M(n,m,\mathbb{F})$, then

$$U(n,\mathbb{F}) = \{X \in GL(n,\mathbb{F}) \mid \bar{X}^t X = I_n\}$$

(I_n the identity matrix), and the orthogonal groups, unitary groups, and symplectic groups are given by $O(n) = U(n,\mathbb{R})$, $U(n) = U(n,\mathbb{C})$, and $Sp(n) = U(n,\mathbb{H})$, respectively. Accordingly, with

$$SU(n,\mathbb{F}) = \{X \in U(n,\mathbb{F}) \mid \det X = 1\}$$

for $\mathbb{F} = \mathbb{R}$ and $\mathbb{C}$ one obtains the special orthogonal and the special unitary groups $SO(n)$ and $SU(n)$, respectively, and with

$$Sp(n,\mathbb{F}) = \{X \in GL(2n,\mathbb{F}) \mid X^t J_n X = J_n\}$$

the real and the complex symplectic groups $Sp(n,\mathbb{R})$ and $Sp(n,\mathbb{C})$. Here J_n denotes the matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Moreover, one has the groups

$$U(n,m,\mathbb{F}) = \{X \in GL(n+m,\mathbb{F}) \mid \bar{X}^t I_{n,m} X = I_{n,m}\}$$

where

$$I_{n,m} = \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}$$

and $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Now polar decomposition provides a retraction from $GL(n,\mathbb{F})$ onto $U(n,\mathbb{F})$ which becomes a strong deformation retract via

$$(X,s) \mapsto X \cdot (\bar{X}^t X)^{-s/2}.$$

Restricting this homotopy to $Sp(n,\mathbb{F})$ and to $U(n,m,\mathbb{F})$ leads to the following deformation retracts

$$Sp(n,\mathbb{R}) \cap O(2n) \cong U(n)$$

$$Sp(n,\mathbb{C}) \cap U(2n) \cong Sp(n)$$

$$U(n,m,\mathbb{F}) \cap U(n+m,\mathbb{F}) \cong U(n,\mathbb{F}) \times U(m,\mathbb{F}).$$

Finally, there are homeomorphisms $U(n) \to SU(n) \times S^1$ and $O(n) \to SO(n) \times \mathbb{Z}_2$ which are both induced by the map

$$X = (X_1,\ldots,X_n) \mapsto ((X_1 \det X^{-1}, X_2, \ldots, X_n), \det X).$$

Therefore, it suffices to consider the compact groups $SO(n)$, $SU(n)$, and $Sp(n)$.

The roots of algebraic topology are usually said to be found in Euler's investigation of the Königsberg bridges problem and in his polyhedron formula. But he may also be considered to be the first who has studied the classical groups when in 1770 he gave a parametrization of the rotation group $SO(n)$ by decomposing each transformation into a product of two-dimensional rotations; of course, he did not yet conceive $SO(n)$ as a topological group. In 1889 Kronecker showed that $O(n)$ consists of two irreducible parts, the fact that constitutes chirality, i.e. the phenomenon of left-handedness and right-handedness in dimension $n = 3$. He also used the alternative parametrization given by Cayley in 1846 to show that $SO(n)$ is an $n(n-1)/2$-dimensional rational manifold. The rotation group as a topological object appeared for the first time in 1897 in a paper by Hurwitz [Htz] where he took up Euler's parametrization to introduce volume elements on $SO(n)$ and on $SU(n)$. The point is a footnote to this construction where he hints at the connectivity of these groups. It was however up to Hermann Weyl to get fully aware of the real importance that the type of connectivity has when one wants to deduce irreducible representations of a Lie group from those of the corresponding Lie algebra.

In 1924 Weyl announced that $SU(n)$ is simply connected and that $SO(n)$ has a two-sheeted simply connected covering group if $n \geq 3$, the so-called spin group $Spin(n)$. It is this double-covering $Spin(3) = SU(2) \to SO(3)$ that underlies the notion of spin in quantum theory distinguishing between fermions and bosons. The complete proofs along with establishing simple connectivity of $Sp(n)$ and finiteness of $\pi_1(G)$ for any Lie group G were presented in his famous series of papers in 1925/6 [Wey1] – that $\mathbb{R}$ covers $SO(2) = S^1$ via the exponential map was already known when Poincaré introduced the first homotopy or fundamental group $\pi_1(X)$ of a topological space X in a short note in 1892. We sketch Weyl's argument for $SU(n)$: Given a closed path $C : [0,1] \to SU(n)$, $C(0) = C(1) = I_n$, consider the "spectral flow" of the matrices $C(t)$, where by a transversality argument if necessary one can assume that all eigenvalues $\lambda_j(t)$, $j = 1, \ldots, n$, $t \neq 0, 1$, are simple. One obtains n closed paths $\lambda_j : [0,1] \to S^1$, which can be chosen to satisfy $\sum \arg \lambda_i(t) = 2\pi$ and $\arg \lambda_1(t) < \cdots < \arg \lambda_n(t)$ for $t \neq 0, 1$. Thus these paths do not cross and are therefore homotopic to the constant path $\lambda(t) \equiv 1$.

Higher homotopy groups had been presented by E. Čech in 1931 at a meeting in Vienna and again in 1932 at the ICM in Zürich but they did not receive much attention until they were rediscovered in 1935 by Hurewicz [Hcz]. In the meantime (in 1928) Elie Cartan [Ctn] had proved that the second homotopy group of a compact Lie group G vanishes. Still written in the language of homology theory (vanishing of Betti numbers) his argument is similar to Weyl's noting that any continuous map $S^2 \to G$ can be deformed to avoid the singular set which has codimension 3 and, therefore, it is homotopic to a constant one.

Further investigations of the homotopical structure of classical groups have been intimately connected with the problem of finding the maximal number of linearly independent vector fields on spheres, in particular, to answer the question of parallelizability. In this context we have to mention the work of Stiefel, Whitney, Eckmann, Ehresmann, Feldbau, Hurewicz, and Steenrod on fibre bundles especially over spheres and Stiefel manifolds continuing previous work of Freudenthal, H. Hopf, and Pontrjagin on the topology of spheres.

The most important new device was the long exact homotopy sequence which

relates the homotopy groups of the total space E, of the base space B, and of the fibre F of a fibre bundle (E, B, F):

$$\to \pi_k(F, *) \to \pi_k(E, *) \to \pi_k(B, *) \to \pi_{k-1}(F, *) \to$$

(here $*$ denote appropriate base points). When applied to the fibration

$$SU(n, \mathbb{F}) \to SU(n, \mathbb{F})/SU(n-1, \mathbb{F}) \cong S^{n\cdot \dim \mathbb{F}-1}$$

it yields that the homotopy groups $\pi_k(SO(n))$, $\pi_k(SU(n))$, and $\pi_k(Sp(n))$ do not depend on n if $n \geq k+2$, $n \geq (k+1)/2$, and $n \geq (k-1)/4$, respectively, since $\pi_k(S^n) = 0$ if $0 \leq k < n$ [Eck1]. We denote these stable homotopy groups by $\pi_k(SO)$, $\pi_k(SU)$, and $\pi_k(Sp)$.

Following Hurwicz' suggestion, Weyl used this method in his book on classical groups [Wey2] to simplify his original computation of the fundamental groups. At this occasion he also gives the following nice argument that shows that the nontrivial loop

$$\phi(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad 0 \leq t \leq 2\pi,$$

in $SO(3)$ when cicumvented twice is homotopic to a constant one: Consider two circular cones both of aperture $\alpha = \frac{\pi}{2}$, one being fixed in space the other rolling on the first. This movemant describes a rotation by 4π about the axis of the moving cone. While increasing the angle α from $\frac{\pi}{2}$ to π the motion of the rolling cone changes to a mere wobble and, finally, at $\alpha = \pi$ comes completely to rest.

The homotopy groups $\pi_k(SO(n))$, $k = 2$, 3, and 4, have been computed independently by Eckmann [Eck2] and G.W. Whitehead [Whi] in 1941. Both also gave $\pi_5(SO(n))$ using a conjecture by Pontrjagin who claimed that $\pi_5(S^3)$ is trivial. But it was only in 1950 that this turned out to be wrong when Pontrjagin and Whitehead found that $\pi_5(S^3) = \mathbb{Z}_2$. The corrected results were presented soon afterwards in a survey article by Eckmann [Eck3] and in Steenrod's book on the topology of fibre bundles [Ste]. Further computation of $\pi_k(SO(n))$ has been done by Borel and Serre ($k = 6$), Serre and Paechter ($k = 7$), Sugawara ($k = 7$, 8, and 9, partial results for $k = 10$ and 11), and Toda [Tod].

However in several places the list given by Toda in 1955 did not agree with results that Borel and Hirzebruch obtained in spring 1957 about the divisibility of certain characteristic classes [BH]. The discussion around this controversy aroused Raoul Bott's interest, and in 1957 [Bot] he found the famous periodicity theorem and confirmed Borel's and Hirzebruch's result.

Theorem *The stable homotopy groups are periodic with*

$$\pi_k(O) \cong \pi_{k+4}(Sp) \cong \pi_{k+8}(SO)$$

and

$$\pi_k(SU) \cong \pi_{k+2}(SU)$$

for $k \geq 0$.

This theorem, a landmark in the history of homotopy theory, did not only help to solve some of the classical problems mentioned before like the parallelizability of spheres (according to Bott, Kervaire, and Milnor only S^1, S^3 and S^7 have trivial tangent bundle), it also provided the fundamentals of K-theory which had been introduced by Atiyah and Hirzebruch. K-theory itself found also many applications; we only mention the solution of the vector field problem by Adams and the index theorem of Atiyah and Singer.

The periodicity theorem determined the homotopy groups of the classical groups in the stable range. Afterwards many topologists went on to explore the metastable range, i.e. $\pi_{2n+k}(SU(n))$, $\pi_{n+k}(SO(n))$, and $\pi_{4n+k}(Sp(n))$ for k small. A lot of computation has been done in the last thirty years above all by Japanese topologists. We do not want to go into details. Instead, we refer to [Lun] which contains tables of the currently known results as well as the credits. We only mention that the methods are essentially to use the exact homotopy sequence applied to fibrations of Stiefel manifolds. The main problem here comes with the 2-primary components, i.e. the subgroups of order a power of two. For a prime $p \geq 3$ the p-primary components π_k^p are much easier to find, for example for the rotation groups one has isomorphisms

$$\pi_k^p(Sp(n)) \cong \pi_k^p(SO(2n+1)), \quad n, k \geq 1.$$

For $n \geq 13$ and $k < n-2$ the homotopy groups $\pi_{n+k}(SO(n))$ can be obtained from those of the Stiefel manifolds

$$St_{n,m}(\mathbb{F}) = \{X \in M(n,m,\mathbb{F}) \mid \bar{X}^t X = I_m\}.$$

One usually denotes the real ones by $V_{n,m}$, the complex ones by $W_{n,m}$, and the quaternionic ones by $X_{n,m}$. The crucial relation found by Barrett and Mahowald in 1964 is then

$$\pi_{n+k}(SO(n)) \cong \pi_{n+k}(SO) \oplus \pi_{n+k+1}(V_{2n,n}).$$

Homotopy groups $\pi_{k+p}(V_{k+m,m})$ are stable for $m \geq p+2$. The metastable range has been explored by Nomura, see [Nom] and references therein for earlier work. Complex Stiefel manifolds have been inspected by Furukawa and Nomura [FN] and $X_{n,2}$ as the only quaternionic Stiefel manifold considered so far by Ôguchi [Ôgu].

Finally, we want to mention the symmetric spaces $SO(2n)/U(n)$, $SU(n)/SO(n)$, $Sp(n)/U(n)$, and $U(2n)/Sp(n)$, and the Grassmannians

$$Gr_{n,m}(\mathbb{F}) = U(n,\mathbb{F})/(U(m,\mathbb{F}) \times U(n-m,\mathbb{F}))$$

which have stable homotopy groups for n large, too. These stable groups also enter Bott's periodicity theorem. More precisely, with the notation of the corresponding stable homotopy groups properly understood one has

$$\pi_k(O) \cong \pi_{k+1}(BO) \cong \pi_{k+2}(U/SO) \cong \pi_{k+3}(Sp/U)$$
$$\pi_k(Sp) \cong \pi_{k+1}(BSp) \cong \pi_{k+2}(U/Sp) \cong \pi_{k+3}(SO/U)$$

and

$$\pi_k(SU) \cong \pi_{k+1}(BU),$$

where BO, BU, and BSp are defined as $\lim Gr_{2n,n}(\mathbb{F})$ with $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, respectively. In the metastable range the homotopy groups can be deduced from appropriate homotopy sequences involving classical groups, see again [Lun] for symmetric spaces; for Grassmannians one uses the following relation due to James [Jam]

$$\pi_k(Gr_{n,m}(\mathbb{F})) \cong \pi_k(St_{n,m}(\mathbb{F})) + \pi_{k-1}(U(m,\mathbb{F})), \quad 2m \leq n.$$

2. The general linear group and the Fredholm group of a Banach space

The possibility to distinguish between different orientations on the Euclidean space $\mathbb{R}^n$ relies on the fact that the orthogonal group $O(n)$ decomposes into exactly two components. To what extent this carries over when one passes to an infinite dimensional real Hilbert space $H_{\mathbb{R}}$ had been asked by Wintner in 1929 after he had shown that the unitary group $U(H_{\mathbb{C}})$ of a complex Hilbert space $H_{\mathbb{C}}$ is connected – by the spectral theorem each unitary operator U is of the form $U = e^{iT}$ with T a self-adjoint operator hence $U(H_{\mathbb{C}})$ allows a parametrization in the sense of Euler-Hurwitz [Win1]. Wintner gave an affirmative answer to this question only about 20 years later in joint work with Putnam [PW], although the prerequisites necessary for the proof were already available in 1932 when Martin proved that any orthogonal transformation T can be written in the form $T = Qe^{-S}$ with S skew-adjoint and $Q = I - 2E$ a symmetry commuting with S. Using this decomposition the proof is easy. Connect e^{-S} with the identity I by the path $t \mapsto e^{-tS}$, $0 \leq t \leq 1$, and then, since $QE = -E$, $Q(I-E) = I - E$, and since at least one of these projections is infinite, connect E with $-E$ or $I - E$ with $E - I$ according to as E or $I - E$ is infinite. To do so and finally to join $-I$ with I one takes an orthonormal basis (e_i) of the corresponding infinite dimensional (sub)space (with obvious modifications in the nonseparable case) and defines a family of orthogonal transformations U_t by

$$\begin{aligned} U_t e_{2i-1} &= e_{2i-1} \cos t - e_{2i} \sin t \\ U_t e_{2i} &= e_{2i-1} \sin t + e_{2i} \cos t\,. \end{aligned}$$

In general Q cannot be connected to I by a path within the set of self-adjoint orthogonal transformations (e.g. if Q has positive and negative eigenvalues) [Win2]. This is in sharp contrast to skew-adjoint orthogonal transformations. Any two $S, T \in O(H_{\mathbb{R}})$ with $T^* = -T$, and $S^* = -S$ belong to one and the same (connected) orbit, since the orthogonal group acts transitively on the set of skew-adjoints by conjugation [Win3]. Indeed, if one takes orthonormal bases (e_i) and (f_i) of $H_{\mathbb{R}}$ with

$$\begin{aligned} T\, e_{2i-1} &= e_{2i}\,, \quad & T\, e_{2i} &= -e_{2i-1}, \\ S\, f_{2i-1} &= f_{2i}\,, \quad & S\, f_{2i} &= -f_{2i-1}\,, \end{aligned}$$

then $Ue_i = f_i$ defines an orthogonal transformation with $U^{-1}SUe_i = Te_i$.

Since any operator in $L(H_{\mathbb{F}})$, the Banach algebra of bounded linear operators on the Hilbert space $H_{\mathbb{F}}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), allows polar decomposition, the same retraction and homotopy as in section 1 prove that the general linear group $GL(H_{\mathbb{F}})$ of invertible operators in $L(H_{\mathbb{F}})$ is connected. This can also be seen in a more elementary way using an idea of Nikolaas H. Kuiper's. Given $T \in U(H_{\mathbb{F}})$ one successively constructs orthogonal unit vectors $a_1, a_2, \dots \in H_{\mathbb{F}}$ and two-dimensional mutually orthogonal subspaces $A_1,\ A_2, \dots$ with $a_i, Ta_i \in A_i$. With H' denoting the closure of the span of the A_i's one defines by

$$S|H'^{\perp} = I_{H'^{\perp}} \ \text{ and } \ S|A_i \text{ a rotation with } S(Ta_i) = a_i$$

an orthogonal transformation that can be connected with the identity, i.e., T can be connected with ST and ST is the identity on the subspace H spanned by the a_i's.

With respect to $H_{\mathbb{F}} = H^{\perp} \oplus H$ one gets $ST = \begin{pmatrix} T'' & 0 \\ T' & I_H \end{pmatrix}$ and the path $\begin{pmatrix} T'' & 0 \\ tT' & I_H \end{pmatrix}$, $0 \le t \le 1$, leads to a diagonal transformation. Now H can be decomposed into countably many mutually orthogonal subspaces isomorphic to $H^{\perp}$ and with respect to this splitting one has

$$T'' \oplus I_H = T'' \oplus T''T''^{-1} \oplus I \oplus T''T''^{-1} \oplus I \cdots$$

Next one connects with

$$T'' \oplus T''^{-1} \oplus T'' \oplus T''^{-1} \oplus T'' \oplus \cdots$$

and then with I where in each step one uses the path

$$t \mapsto \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & I \end{pmatrix},\ 0 \le t \le \frac{\pi}{2},$$

between

$$\begin{pmatrix} uv & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}.$$

In 1963 Shvarts and Jänich noticed that this trick can be used to contract the stable group $GL(\infty, \mathbb{F}) = \lim\limits_{\rightarrow} GL(n, \mathbb{F})$ or even the Fredholm group $GL_c(H_{\mathbb{F}}) = \{T \in GL(H_{\mathbb{F}}) \mid T - I \text{ compact}\}$ to a point within $GL(H_{\mathbb{F}})$. Shvarts, Palais, and Atiyah then conjectured that already $GL(H_{\mathbb{F}})$ should be contractible, and in due course a proof was provided by Kuiper [Kui] using a refinement of the previous idea. We cannot dwell on the consequences of this far-reaching result, e.g. it implies that any Hilbert space bundle over a paracompact space is trivial and that the unit sphere of Hilbert space is parallelizable. Contractibility of $GL(H_{\mathbb{F}})$ in the strong topology had been proved before by Douady and Dixmier [DD].

Of course, the question immediately came up to what extent Kuiper's result carries over to arbitrary Banach spaces, since already Shvarts had noticed that for some Banach spaces (among others l^p, $1 \le p \le \infty$, and c_0) the Fredholm group $GL_c(E)$ is contractible within $GL(E)$ (see also [Deu2] for a recent variant). In 1965 Arlt proved that c_0, the space of sequences tending to zero, is a "Kuiper space", but Douady found examples of complex Banach spaces whose general linear groups are not even connected. Specifically, he proved the following result, cf. the survey paper [Mit].

Theorem 1 *If E and F are Banach spaces both isomorphic to their hyperplanes and if $L(E, F) = L_c(E, F)$, then $GL(E \times F)$ has the homotopy type of*

$$GL(E) \times GL(F) \times \mathbb{Z} \times BGL(\mathbb{F}),$$

where $BGL(\mathbb{F}) = \lim\limits_{\rightarrow} GL(2n, \mathbb{F})/(GL(n, \mathbb{F}) \times GL(n, \mathbb{F}))$.

Since then many Banach spaces have been inspected. The classical sequence spaces l^p, $1 \le p < \infty$, (and again c_0) were shown to be Kuiper spaces by Neubauer. On the other hand, l^p for $0 < p < 1$ is only a metrizable topological vector space not even locally convex but applying Neubauer's methods Cuellar [Cue] found that the homotopy groups of the general linear group also vanish while it is still not known

whether it is contractible. Neubauer's method has been refined by Mitjagin and his coworkers leading to the following sufficient criterion for contractibility.
A Banach space E is called weakly infinitely decomposable (WID) if $E \cong E \times E$, if there exists a total family of mutually orthogonal projections P_k and isomorphisms $T_k : P_k E \to E$, $k \geq 0$, such that

$$\begin{aligned} T_k P_k S &= T_{k+1} P_{k+1}, k \geq 0, \\ T_k P_k S' &= T_{k-1} P_{k-1}, k \geq 1, \\ P_0 S' &= 0 \end{aligned}$$

for some $S, S' \in L(E)$, and if for any $B \in L(E)$ there exists $\tilde{B} \in L(E)$ with

$$T_k P_k \tilde{B} = B T_k P_k, \quad k \geq 0.$$

E has the property of smallness of operator blocks (SOB) if for any $\epsilon > 0$ and for any compact family of operators, $\{B\}$, there are orthogonal projections Q_1, Q_2 with $Q_i E \cong E$, $i = 1, 2$, and $\|Q_1 B Q_2\| \leq \epsilon$, for all B.

Theorem 2 *If E has properties WID and SOB then $GL(E)$ is contractible.*

Using this general condition the following Banach spaces can be shown to be Kuiper spaces, cf. [Mit] for proofs or references:
$C(K, F)$ where F is a Banach space not having c_0 as a direct summand, and where K is either
– an uncountable compact metric space,
– an infinite compact topological group,
– an infinite product of non-one-point compact metric spaces,
– the Stone space of an infinite homogeneous measure algebra,
– or the Stone-Čech compactification of the integers.
This also holds if K is the product of τ copies of a two-point space with the topology induced by lexicographic order where $\tau \geq \omega$ is a countable ordinal, cf. [Sem].
Moreover, one has contractibility for
– $C^k(M)$, the space of k-times differentiable functions on a smooth compact manifold M,
– the Sobolev spaces $H^p_s(D)$, $1 < p < \infty$, $s \geq 0$, and $W^p_\ell(D)$, $1 < p < \infty$, $\ell \in \mathbb{N}$, on a domain $D \subset \mathbb{R}^n$ with regular boundary,
– the spaces $L^p([0,1])$, $1 \leq p \leq \infty$, of p-integrable measurable functions,
– the spaces $L^p(\Omega, \mu)$, $1 < p < \infty$, of p-integrable measurable functions on an arbitrary measure space Ω,
and for some classes of reflexive symmetric function spaces.
Another class of Banach spaces with contractible general linear group consists of operator algebras, e.g. $L(H)$, H a Hilbert space, or, more generally, $L(\mathcal{M})$, $\mathcal{M}$ an injective von Neumann factor of infinite type or the hyperfinite factor of tye II_1 [SS], a countable decomposable von Neumann factor of type III [Wil] and [SS], or $C_p(H)$, $1 \leq p \leq \infty$, the von Neumann-Schatten classes.

There are also more non-Kuiper spaces. For example, Douady's theorem applies to the following products $E \times F$ of Banach spaces: $c_0 \times l^p$, $l^{p_1} \times l^{p_2}$, $1 \leq p_1 < p_2 < \infty$, and $J^{n_0}(p_0, \mathbb{F}) \times J^{n_1}(p_1, \mathbb{F})$, $1 \leq p_0 < p_1 < \infty$, where $J^m(p, \mathbb{F})$ is the m^{th} power

of the James space $J(p, \mathbb{F})$. More precisely, $GL(J^m(p, \mathbb{F}))$ is homotopy equivalent to $GL(m, \mathbb{F})$ as is $GL(C(\Gamma_{m\omega_1}, \mathbb{F}))$ where $\Gamma_{m\omega_1}$ denotes the "long line" of ordinals $\leq m\omega_1$, ω_1 the first uncountable ordinal [Bel]. Finally, we mention that the general linear group $GL(X)$ of a real or complex nuclear Fréchet space X with basis is dense in the algebra of bounded operators $L_b(X)$ and is connected in the induced topology [Gra].

So far, we have only considered the general linear group of a Banach space. Other "classical" groups can be defined as well. In order to define orthogonal, unitary, or symplectic groups one needs, respectively, a nondegenerate symmetric, hermitean, or skew-symmetric bilinear form φ on a real (complex) Banach space E which is then necessarily reflexive [Swa]. The subgroups of $GL(E)$ whose elements are φ-invariant are real Banach Lie groups. Here we stick to Hilbert spaces, but see [Swa] for symplectic groups on Banach spaces. We have already met the orthogonal and unitary groups, $O(H_\mathbb{R})$ and $U(H_\mathbb{C})$, which are deformation retracts of $GL(H_\mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The analogues of $U(n, m, \mathbb{F})$ and $Sp(n, \mathbb{R})$ are the group of Q-unitary operators

$$U(Q, H_\mathbb{F}) = \{T \in L(H_\mathbb{F}) \mid T^*QT = Q = TQT^*\},$$

where $Q = Q^* = Q^{-1} = 2P - 1 \in GL(H_\mathbb{F})$, and the real symplectic group

$$Sp(J, H_\mathbb{R}) = \{T \in L(H_\mathbb{R}) \mid T^*JT = J = TJT^*\},$$

where $J \in O(H_\mathbb{R})$, $J^2 = -I$. The homotopy type of these groups is more or less folklore and is in fact easily deduced from Kuiper's theorem. That $U(Q, H_\mathbb{F})$ is connected has been noted in 1968 by Phillips [Phi], but was most likely already known to Wintner in the thirties. The same homotopy as in section 1 shows that $U(Q, H_\mathbb{F})$ retracts onto $U(PH_\mathbb{F}) \times U((1-P)H_\mathbb{F})$ [Kuč], and that $Sp(J, H_\mathbb{R})$ is contractible [Swa].

When not imposing an additional structure on a Banach space E given by a bilinear form one might think of the group of invertible isometries $Iso(E)$ as the appropriate substitute for the group of unitaries. But this group may have a rather peculiar topological structure and may even fail to be a Banach Lie group [HK]. However, one has the result by Stern [Str] that there are real Banach spaces whose group of invertible isometries is isomorphic to any given group G that contains a normal subgroup with two elements e and τ. More precisely, he proved that there exists a real Banach space E and an isomorphism $\Phi : E \to H$ with $\|\Phi\|\|\Phi^{-1}\| \leq 1 + \epsilon$ onto a Hilbert space H such that $Iso(E) \cong G$ with e corresponding to Id_H and τ to $-Id_H$. In particular, $GL(E)$ is contractible while $Iso(E)$ is any group extension by $\mathbb{Z}_2$. If E is a complex Banach space each component of $Iso(E)$ contains of course at least a circle.

After this digression and before we come to the Fredholm group we take a brief look at Stiefel and Grassmann manifolds in infinite dimensions. The Grassmann manifold $Gr(E)$ of an infinite dimensional Banach space E is the set of complemented subspaces of E with the opening topology (for not necessarily complemented closed subspaces see [CrL]). The subset of complemented subspaces isomorphic to a given complemented subspace $F \subset E$ is denoted by $Gr_F(E)$. Specifically, we denote by $Gr_n(E)$ and $Gr_{\infty-n}(E)$ the set of n-dimensional and n-codimensional subspaces of E, respectively. The set $St_F(E)$ of monomorphisms of a complemented subspace F into E with complemented range generalizes the Stiefel manifold in infinite dimensions. $GL(F)$ acts on $St_F(E)$ and $Gr(E) = St_F(E)/GL(F)$. If F is an n-dimensional subspace of E then $St_F(E)$ is contractible [DD], and hence the exact homotopy sequence

of fibre bundles determines the homotopy type of $Gr_n(E)$ and of $Gr_{\infty-n}(E)$. In case of a Hilbert space E the various homotopy types are completely known, cf. [Luf1]; in general, they depend on the homotopy type of $GL(E)$.

Whereas $GL_c(E)$ is contractible within $GL(E)$ for some Banach spaces E the Fredholm group $GL_c(E)$ itself is homotopy equivalent to $GL(\infty, \mathbb{F})$. This was shown by Shvarts and Palais in 1963 for a Hilbert space and in general independently by Elworthy [ET] and Geba [Geb] in 1968. Already Palais had considered the groups $GL_{C_p}(H_{\mathbb{F}})$ defined analogously using the symmetric ideals $C_p(H_{\mathbb{F}})$ of von Neumann-Schatten operators instead of the ideal of compacts $C(H_{\mathbb{F}})$. These groups are again homotopy equivalent to $GL(\infty, \mathbb{F})$. Geba's proof implies the following more general result stated in [dlH1]:

Theorem 3 *If E is a Banach space and $P(E) \subset L(E)$ a normed (not necessarily closed) subspace with*
i) $1 + X \in Fred(E)$ (= set of Fredholm operators)
ii) $X + C_0(E) \subset P(E)$ ($C_0(E)$ the ideal of finite rank operators) for all $X \in P(E)$ and
iii) $C_0(E) \times P(E) \ni (T, X) \mapsto TX \in P(E)$ continuous,
then $GL_p(E) = \{1 + X \in GL(E) \mid X \in P(E)\}$ is homotopy equivalent to $GL(\infty, \mathbb{F})$.

In particular, $GL_{C_p}(H_{\mathbb{R}})$ and its deformation retract $O_{C_p}(H_{\mathbb{R}})$ have two connected components and the component $SO_{C_p}(H_{\mathbb{R}}) \subset O_{C_p}(H_{\mathbb{R}})$ that contains the identity has a two-sheeted universal covering group $Spin_{C_p}(H_{\mathbb{R}})$, which can be realized as a subgroup of the unitary group of the infinite dimensional Clifford C^*-algebra in case that $p = 1$ [dlH2], and of the hyperfinite W^*-algebra factor of type II_1 in case that $p = 2$ [Ply].

Moreover, for $J \in O(H_{\mathbb{R}})$ with $J^2 = -I$, $J - J^* \in \mathcal{S}$ one defines

$$G_{C_p}(H_{\mathbb{R}}, J) = \{T \in GL(H_{\mathbb{R}}) \mid TJ - JT \in C_p(H_{\mathbb{R}})\}$$

with retract

$$O_{C_p}(H_{\mathbb{R}}, J) = G_{C_p}(H_{\mathbb{R}}, J) \cap O(H_{\mathbb{R}}) \cong O/U,$$

and for $Q \in U(H_{\mathbb{F}})$, $Q = Q^*$, the group

$$G_{C_p}(H_{\mathbb{F}}, Q) = \{T \in GL(H_{\mathbb{F}}) \mid TQ - QT \in C_p(H_{\mathbb{F}})\}.$$

They appear as regular groups of the Banach algebras of operators $T \in L(H_{\mathbb{F}})$ with

$$\|T\|_{p,V} = \|T\| + \|TV - VT\|_{\mathcal{S}} < \infty,$$

where $V = J$ or Q, cf. [CE] and previous work by Carey and coauthors cited there. $G_{C_1}(H_{\mathbb{C}}, Q)$ was denoted by $G_{res}(H)$ in [PrS], cf. also [Woj]. More generally, Carey and Evans [CE] considered Banach algebras consisting of operators that commute with up to one generator of a Clifford algebra $\mathcal{C}_k$, the commutator with the last generator lying in a symmetric ideal $\mathcal{S}$ of compact operators. The periodicity of the Clifford algebras (of period 8 in the real and of period 2 in the complex case) is inherited by these algebras and the principal components of the corresponding regular groups have the same homotopy type as the stablilized homogeneous spaces encountered at the end

of section 1. Moreover, suitably modified they can be used as classifying spaces in KK-theory.

The classical example of a classifying space is the set $Fred(H_{\mathbb{F}})$ of Fredholm operators which is the preimage of the regular group of the Calkin algebra $Cal(H_{\mathbb{F}}) = L(H_{\mathbb{F}})/C(H_{\mathbb{F}})$ under the canonical quotient map π. Here π induces even a homotopy equivalence between $Fred(H_{\mathbb{F}})$ and $G(Cal(H_{\mathbb{F}}))$, cf. the appendix in [Ati]. The components of $Fred(H_{\mathbb{F}})$ are separated by the index map which indeed is an isomorphism between the set of components and $\mathbb{Z}$. This isomorphism has been generalized by Atiyah and Singer [AS] and by Karoubi [Kar2] as follows:

Theorem 4 *For any compact Hausdorff space X, and $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ one has*

$$K^n_{\mathbb{F}}(X) \cong [X, F^n(H_{\mathbb{F}})],$$

where $F^0(H_{\mathbb{F}}) = Fred(H_{\mathbb{F}})$ and where $F^n(H_{\mathbb{F}})$ is a subset of $Fred(H_{\mathbb{F}})$ characterized by certain spectral properties.

For equivariant K-theory $K^n_G(X)$, G a compact Hausdorff group, one has an isomorphism of $K^n_G(X)$ with $[X, F^n(H_{\mathbb{F}})]_G$ when $H_{\mathbb{F}}$ is a G-module and $F^n(H_{\mathbb{F}})$ is considered as a G-space ($[\cdot,\cdot]_G$ denotes G-homotopy classes of G-invariant maps). This is proved for $n = 0$ by Segal and Matumoto and in general by Kučment and Pankov [KP] using an equivariant version of Kuiper's theorem given by Segal. Note that in these papers it is tacitly assumed that the G action is norm continuous. While the equivariant contractibility is by the time still open the above isomorphism nonetheless holds, cf. [Phs2].

There are partial extensions to Fredholm operators (or semi-Fredholm operators) on a Banach space E, see [ZKKP] for the following and for further references. Here one has an exact sequence

$$[X, GL_c(E)] \to [X, GL(E)] \to [X, Fred(E)] \to K^0_{\mathbb{F}}(X),$$

valid for any compact Hausdorff space X. If E contains a complemented infinite dimensional subspace with symmetric basis then this leads to the exact sequence

$$0 \to [X, GL(E)] \to [X, Fred(E)] \to K^0_{\mathbb{F}}(X) \to 0$$

and in case that E is a Kuiper space to the isomorphism

$$K^0_{\mathbb{F}}(X) \cong [X, Fred(E)].$$

If E and F are infinite dimensional Banach spaces and if the set $Fred(E, F)$ of Fredholm operators in $L(E, F)$ is nonempty then $GL(E)$ is connected (contractible) if and only if $GL(F)$ is. Moreover, in this case the components of $Fred(E, F)$ consist just of Fredholm operators with the same index [GK].

Also the Calkin algebra has been inspected in more detail. The homotopy type of the regular group is completely determined by the previous theorem. The set of projections consists of three arcwise connected components, $\{0\}$, $\{1\}$, and the set of nontrivial projections. Each component is simply connected, and so is each component of the regular group [KK].

3. General Banach algebras

Let $\mathcal{B}$ be a real or complex Banach algebra with unit and let $G(\mathcal{B})$ denote the group of invertible elements. $G(\mathcal{B})$ is a Banach Lie group and an open subset of $\mathcal{B}$. Therefore, $G(\mathcal{B})$ is locally arcwise connected and its components are arcwise connected. We denote the principal component, that is the one which contains the unit element, by $G^0(\mathcal{B})$. $G^0(\mathcal{B})$ is a Banach Lie group, too, and a normal subgroup of $G(\mathcal{B})$; the quotient group $I(\mathcal{B}) = G(\mathcal{B})/G^0(\mathcal{B})$ is called the index group of $\mathcal{B}$. Besides the regular group $G(\mathcal{B})$ we also consider the set

$$Id(\mathcal{B}) = \{p \in \mathcal{B} \mid p^2 = p\}$$

of all idempotent elements which is homeomorphic to the set of symmetries

$$Sym(\mathcal{B}) = \{x \in \mathcal{B} \mid x^2 = 1\}.$$

If $\mathcal{B}$ is real one also has

$$D(\mathcal{B}) = \{x \in \mathcal{B} \mid x^2 = -1\}.$$

It has been shown by various authors that $Id(\mathcal{B})$ is locally arcwise connected in the relative topology and that the components are arcwise connected. More precisely, the components are equal to the orbits of $G^0(\mathcal{B})$ acting on $Id(\mathcal{B})$ by conjugation, cf. the references in [CPR]. The same holds for the action of $G^0(\mathcal{B})$ on $D(\mathcal{B})$ [Fuj2]. Corach, Porta, and Recht [CPR] give a more general result: If $P(x)$ is a polynomial with simple roots then

$$A = \{a \in \mathcal{B} \mid P(a) = 0\}$$

is locally arcwise connected and the component containing a is just again the orbit $\{gag^{-1} \mid g \in G^0(\mathcal{B})\}$.

On $Id(\mathcal{B})$ one defines an equivalence relation by

$$p \sim q \quad \text{iff} \quad pq = q \quad \text{and} \quad qp = p.$$

The quotient space $Gr(\mathcal{B}) = Id(\mathcal{B})/_\sim$ is called the Grassmannian of $\mathcal{B}$. The main result here is [PR1]:

Theorem 1 *$Gr(\mathcal{B})$ and $Id(\mathcal{B})$ are homotopy equivalent.*

Because given $p \in Id(\mathcal{B})$ one can find a neighborhood U with $1-(q-p)^2$ invertible for $q \in U$ and $q \mapsto [1-(q-p)^2]^{-1}q$ continuous on U. Then $h : U \ni q \mapsto [1-(q-p)^2]^{-1}qp \in Id(\mathcal{B})$ is a continuous map. Using a partition of unity one defines h globally. Now $h : Id(\mathcal{B}) \to Id(\mathcal{B})$ is continuous and has the following properties

$$\begin{aligned} ph(p) &\sim p \quad \text{for all} \quad p, \\ h(p) = h(q) &\quad \text{iff} \quad p \sim q. \end{aligned}$$

The same proof carries over to topological algebras with open regular group, continuous inversion, and $Gr(\mathcal{B})$ paracompact [PR2], cf. also the paper [Grm] by Gramsch where these results are obtained by explicit calculations using “rational coordinates”. If $\mathcal{B}$

is a C^*-algebra, $Gr(\mathcal{B})$ can be identified with the set of projections, i.e. self-adjoint idempotents.

Abstract Banach algebras had been introduced in 1936 by Nagumo as linear metric rings. In 1939 Gelfand founded the theory of maximal ideals for commutative Banach algebras and showed that the set of all maximal ideals can be endowed with a topology making it into a locally compact Hausdorff space which is compact if the algebra has a unit. If we identify maximal ideals with kernels of characters, i.e., nontrivial homomorphisms $\varphi : \mathcal{B} \to \mathbb{C}$, this topology is nothing but the weak-$*$-topology on the set of characters. Via the correspondence between the space of maximal ideals $M(\mathcal{B})$ and the set of characters $\Omega(\mathcal{B})$ each $x \in \mathcal{B}$ defines a continuous function $\hat{x} \in C(M(\mathcal{B}))$ by $\hat{x}(\omega) = \omega(x)$, $\omega \in \Omega(\mathcal{B})$. The assignment

$$\mathcal{B} \ni x \mapsto \hat{x} \in C(M(\mathcal{B})),$$

the so-called Gelfand transform, is a homomorphism which in general is neither injective nor surjective.

Even in the commutative case investigations of the homotopy type of the regular group of a Banach algebra have not been taken further than to determine the index group and the fundamental group of the principal component. The first algebra under inspection was the algebra $CAP(\mathbb{R})$ of continuous almost periodic functions on $\mathbb{R}$ (here $M(CAR(\mathbb{R})) = \mathbb{R}^B$, the so-called Bohr compactification of $\mathbb{R}$). In 1930 H. Bohr proved a conjecture by Wintner who had claimed that the regular group of $CAP(\mathbb{R})$ has uncountably many components labeled by the "mean motion" or "mean winding number". For $f \in CAP(\mathbb{R})$ this winding number is defined by

$$\tau(f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} d(arg\, f)$$

and generalizes the usual winding number $\tau(f)$ of a periodic function f defined by

$$\tau(f) = \frac{1}{2\pi i} \int_{S^1} f^{-1} df = \frac{1}{2\pi} \int_0^{2\pi} d(arg\, f).$$

Both maps induce isomorphisms $I(G(CAP(\mathbb{R}))) \cong \mathbb{R}$ and $I(C(S^1)) \cong \mathbb{Z}$. In 1936 van Kampen has extended this result to almost periodic functions on a connected Lie group G, proving that to each connected component of the regular group $G(CAP(G))$ there corresponds a group character. Since $CAP(S^1) = C(S^1)$, one recovers once again the classical result $I(C(S^1)) \cong \mathbb{Z}$.

For a compact metric space X Bruschlinsky had already shown in 1934 that the index group of $C(X)$ coincides with its first Čech cohomology group $H^1(X, \mathbb{Z})$, and so contains in particular no element of finite torsion, a fact established by Lorch in 1942 for any commuative Banach algebra. That $I(\mathcal{B})$ is commutative for any commutative Banach algebra has been proved independently in 1960 by Royden and Arens; more precisely:

Theorem 2 *For a commutative Banach algebra $\mathcal{B}$ with unit one has*

$$I(\mathcal{B}) \cong H^1(M(\mathcal{B}), \mathbb{Z}).$$

For noncommutative Banach algebras this turns out to be wrong in general, $I(\mathcal{B})$ may be neither commutative nor torsion free. To see this consider the index group of $\mathcal{B} = C(S^k, M(n,\mathbb{C}))$ which is isomorphic to $[S^k, U(n)]$, and hence

$$I(\mathcal{B}) \cong \pi_k(U(n))$$

since $U(n)$ is arcwise connected. This gives finite nontrivial index groups for suitable n and k, cf. the references in section 1. A simpler example of a finite (commutative) index group has been found by Paulsen [Pau]:

$$SM(n,\mathbb{C}) = \{f \in C([0,1], M(n,\mathbb{C})) \mid f(0), f(1) \in \mathbb{C}I_n\},$$

the so-called unreduced suspension of $M(n,\mathbb{C})$, has index group isomorphic to $\mathbb{Z}_n$. A nonabelian index group is obtained for $\mathcal{B} = C(U(2) \times U(2), M(2,\mathbb{C}))$. Here

$$I(\mathcal{B}) \cong [U(2) \times U(2), U(2)]$$

is nonabelian since according to I.M. James the multiplication in $U(2)$ is not homotopy commutative, i.e.

$$(X,Y) \mapsto XY \quad \text{and} \quad (X,Y) \mapsto YX$$

are not homotopic [Yue].

It is an open problem to determine the groups that can be realized as the index group of some Banach algebra or that arise as index group of the operator algebra $L(E)$ for some Banach space E.

The fundamental group of the principal component of a commutative Banach algebra $\mathcal{B}$ is isomorphic to the kernel of the exponential map $\exp : \mathcal{B} \to G^0(\mathcal{B})$ which is onto and induces a covering map as in the classical case $\mathcal{B} = \mathbb{C}$. Combining this observation by Blum from 1952 [Blu] with Shilov's decomposition theorem that characterizes the idempotents of $\mathcal{B}$ by the closed and open sets in $M(\mathcal{B})$ one obtains

Theorem 3 *The fundamental group $\pi_1(G^0(\mathcal{B}))$ of the principal component $G^0(\mathcal{B})$ of a unital commutative Banach algebra $\mathcal{B}$ is isomorphic to $H^0(M(\mathcal{B}), \mathbb{Z})$.*

So far we have assumed without saying that $\mathcal{B}$ is a complex commutative Banach algebra. Indeed, the structure of real commutative Banach algebras is more complicated. For example, a real commutative C^*-algebra is in general not of the form $C(X, \mathbb{R})$, but isomorphic to

$$C(X,\tau) = \{f \in C(X) \mid \overline{f(x)} = f(\tau x)\,,\ x \in X\}\,,$$

where τ is an involutive homeomorphismus of X. Accordingly, the real analogues of the Arens-Royden theorem proved by Alling and Campbell [AC] in 1971 and by Furutani [Frt] in 1975 are more involved.

Theorem 4 *Let $\mathcal{B}^\sharp$ denote the set of all $a \in G(\mathcal{B})$ such that to each $N \in M(\mathcal{B})$ with $\mathcal{B}/N \cong \mathbb{R}$ there exists $a_N > 0$ with $a - a_N \in N$. Then $\mathcal{B}^\sharp/\exp\mathcal{B}$ is isomorphic to $H^1(M(\mathcal{B}), \mathbb{Z}^G)$, the first cohomology group of $M(\mathcal{B})$ with coefficients in a certain sheaf of abelian groups $\mathbb{Z}^G$.*

The simpler results of Lorch and Blum had already been carried over by Ingelstam [Ing] in 1964. For a real commutative Banach algebra $\mathcal{B}$, not containing a complex subalgebra, he proved that $I(\mathcal{B})$ is either infinite or isomorphic to the group of idempotents and that $G^0(\mathcal{B})$ is simply connected in this case.

For quite a time it has been tried to bring also the higher Čech homology groups into the game, but successfully so far only for $k \leq 3$ [Tay3]. A different direction has been taken by Arens in 1965 when he looked at the regular group $GL(n, \mathcal{B})$ of the tensor product $\mathcal{B} \otimes M(n, \mathbb{C}) = M(n, \mathcal{B})$, cf. the survey articles by Forster [For] and by Taylor [Tay1]. Using a deep theorem of Grauert from complex analysis he found that

$$[GL(n, \mathcal{B})] \cong [M(\mathcal{B}), GL(n, \mathbb{C})] \, .$$

The relation between $GL(n, \mathcal{B})$ and $C(M(\mathcal{B}), GL(n, \mathbb{C}))$ has been studied further by Eidlin and Novodvorskii. In particular, Novodvorsikii's methods have been exploited later by Taylor [Tay2] and Raeburn [Rae] who proved the following result.

Theorem 5 *Let F be a closed Banach submanifold of an open subset in a Banach space E. If F is a discrete union of Banach homogeneous spaces then*

$$\mathcal{B}_F := \{a \in \mathcal{B}\hat{\otimes} E \mid a = Sf \, , \; f \in \mathcal{O}(U, F) \, , \; U \supset M(\mathcal{B})\}$$

(where S is the extension of functional calculus and $\mathcal{O}(U, F)$ is the set of analytic functions from U to F) is locally arcwise connected and the Gelfand transform induces a bijection

$$[\mathcal{B}_F] \to [M(\mathcal{B}), F] \, .$$

All of the previous results can be obtained by specialization, e.g. Arens' result by taking $F = GL(n, \mathbb{C}) \subset \mathbb{C}^{n^2}$. For $E = \mathbb{C}^n$ the theorem had been proved in [Tay2]. In this special case $\mathcal{B}_F$ can be replaced by the homotopy equivalent set

$$\mathcal{B}^F = \{a = (a_1, \cdots, a_n) \in \mathcal{B}^n \mid \sigma(a) \subset F\}$$

where $\sigma(a)$ denotes the Taylor spectrum of a [LZ]. Several corollaries are stated in [Tay2], [Rae], and in papers by Fujii [Fuj1-3]. For example, if $E = \mathcal{A}$ is a unital Banach algebra and $F = G(\mathcal{A})$ Raeburn recovers a theorem by Davie which even holds for Fréchet algebras.

Corollary 1 *If $\mathcal{A}$ is a unital Banach algebra then the Gelfand transform induces a homotopy equivalence between $G(\mathcal{B}\hat{\otimes}\mathcal{A})$ and $C(M(\mathcal{B}), G(\mathcal{A}))$.*

Analogous results can be obtained taking $F = Id(\mathcal{A})$ [Rae], $D(\mathcal{A})$ [Fuj2], or $Gr(\mathcal{A})$ [PR1].

Since we are mostly interested in higher homotopy groups $\pi_k(G^0(\mathcal{B}))$ we take $E = C(S^k)$ and get a homotopy equivalence between

$$G(\mathcal{B}\hat{\otimes} C(S^k)) = G(C(S^k, \mathcal{B})) = C(S^k, G(\mathcal{B}))$$

and

$$\begin{aligned} C(M(\mathcal{B}), G(C(S^k))) &= G(C(M(\mathcal{B}), C(S^k))) \\ &= G(C(M(\mathcal{B}) \times S^k)) \\ &= C(M(\mathcal{B}) \times S^k, \mathbb{C} \setminus \{0\}), \end{aligned}$$

Thus the computation of $\pi_k(G^0(A))$ is reduced to the computation of the cohomotopy groups $\pi^1(M(\mathcal{B}) \times S^k)$ – a by no means easier problem. It seems to become even more difficult if one takes $E = C(S^k, M(n, \mathbb{C}))$ instead. However, passing to the direct limit $\lim\limits_{n\to\infty} C(S^k, M(n, \mathbb{C}))$ leads into the field of K-theory and simplifies in the following result first obtained by Novodvorskii and later by Taylor.

Corollary 2 *For any complex commutative Banach algebra $\mathcal{B}$ one has*

$$K_i(\mathcal{B}) \cong K^{-i}(M(\mathcal{B})),\ i = 0, 1.$$

Here $K_i(\mathcal{B})$ is the K-theory of a Banach algebra defined as classes of stable equivalent finitely generated projective $\mathcal{B}$-modules if $i = 0$ and as $\pi_0(GL(\infty, \mathcal{B}))$ if $i = 1$, cf. [Tay1], and $K^{-i}(M(\mathcal{B}))$ is the topological K-theory of the compact space $M(\mathcal{B})$, cf. [Ati] and [Kar3]. In particular, the Chern character of K-theory and "rationalization" lead to a weak characterization envisaged earlier involving higher Čech cohomology groups, since

$$K^{-i}(M(\mathcal{B})) \otimes \mathbb{Q} \cong \bigoplus_{k \geq 0} H^{2k+i}(M(\mathcal{B}), \mathbb{Q}).$$

The K-groups of some topological spaces are known but we can neither discuss the techniques to obtain them nor display the results since this would be beyond the scope of our survey (some special examples will appear in the next section).

K-theory for Banach algebras is not restricted to the commuative case. Most of the theory in [Tay1] is also valid for general Banach algebras (even graded [vDa] and [Kan]) or Fréchet algebras [Phs1]. The fundamental result is the generalized periodicity theorem proved by Wood in 1965 [Woo], cf. also [Kar1,3].

Theorem 6 *If $\mathcal{B}$ is a unital Banach algebra then*

$$K_0(\mathcal{B}) \cong \pi_{2k-1}(GL^0(\infty, \mathcal{B}))\ \textit{and}\ K_1(\mathcal{B}) \cong \pi_{2k}(GL^0(\infty, \mathcal{B})),\ k \geq 1,$$

where $GL^0(\infty, \mathcal{B})$ denotes the principal component of $GL(\infty, \mathcal{B})$.

There are more parallels to the homotopical structure of the classical groups, at least in special cases: if the Bass stable rank $sr(\mathcal{B})$ of $\mathcal{B}$ is finite according to Corach and Larotonda (see [CoL], [Cor], and [CoS2]) the homotopy groups $\pi_k(GL(n, \mathcal{B}))$ stabilize for large n. Before we state the proper result, recall that the Bass stable rank $sr(\mathcal{B})$ is the smallest number n (or ∞ if no such n exists) such that for all $(a_1, \cdots, a_{n+1}) \in \mathcal{B}^{n+1}$ with $\sum a_i \mathcal{B} = \mathcal{B}$ there is a $(b_1, \cdots, b_n) \in \mathcal{B}^n$ with $\sum (a_i + b_i a_{n+1}) \mathcal{B} = \mathcal{B}$. For $\mathcal{B} = C(X, \mathbb{F})$ one has $sr(\mathcal{B}) = [\frac{1}{\dim \mathbb{F}} \dim X] + 1$ and if $\mathcal{B}$ is a C^*-algebra then $sr(\mathcal{B})$ coincides with the topological stable rank $tsr(\mathcal{B})$ introduced by Rieffel [Rie1]. $tsr(\mathcal{B})$ is the smallest number n such that the set of n-tuples of elements of $\mathcal{B}$ which generate $\mathcal{B}$ as a left (right) ideal is dense in $\mathcal{B}^n$ or in case of a C^*-algebra that $R_n(\mathcal{B}) = \{x \in \mathcal{B}^n \mid \sum x_i^* x_i \in G(\mathcal{B})\}$ is dense in $\mathcal{B}^n$. Note that $sr(\mathcal{B})$ and $tsr(\mathcal{B})$ may differ for

a commutative Banach algebra $\mathcal{B}$, e.g. the polydisc-algebra $A(\mathbb{D}^n)$ has stable rank $[\frac{n}{2}]+1$ and topological stable rank $n+1$, cf. [CoS1].

Theorem 7 *If $\mathcal{B}$ is a unital Banach algebra (or a topological algebra with open regular group) with finite stable rank then*

$$\pi_k(GL(n,\mathcal{B})) \cong \pi_k(GL(n+1,\mathcal{B}))$$

and

$$\pi_k(Id(M(n,\mathcal{B}))) \cong \pi_k(Id(M(n+1,\mathcal{B})))$$

if $n \geq sr(\mathcal{B})+k+1$.

If $sr(\mathcal{B})$ is not finite both assertions may be wrong for any n. Already the index group $I(M(n,\mathcal{B})) = \pi_0(GL(n,\mathcal{B}))$ may be unstable in this case, cf. [Srd4].

Finally, we want to look at Banach algebras $\mathcal{A}$ and $\mathcal{B}$ which are related by a dense injective morphism $\varphi : \mathcal{A} \to \mathcal{B}$ and see how this influences the homotopy type of the corresponding regular groups. First we have Karoubi's density theorem [Kar4] (which in a special case goes back to Atiyah [Ati]).

Theorem 8 *If $\varphi : \mathcal{A} \to \mathcal{B}$ is a dense injective morphism of unital Banach algebras, i.e., $\varphi(\mathcal{A})$ is dense in $\mathcal{B}$, and if*

$$GL(n,\mathcal{B}) \cap M(n,\mathcal{A}) = GL(n,\mathcal{A})$$

for any $n \geq 1$ (identifying $\mathcal{A}$ with its image), then φ induces isomorphisms

$$\varphi_* : K_i(\mathcal{A}) \to K_i(\mathcal{B}),\ i \geq 0.$$

In particular, this is guaranteed for complex Banach algebras if $\mathcal{A}$ is stable under holomorphic functional calculus in $\mathcal{B}$, i.e., if for any n and $a \in M(n,\mathcal{A}) \subset M(n,\mathcal{B})$ one has $f(a) \in M(n,\mathcal{A})$ for f holomorphic in a neighborhood of the spectrum of a in $M(n,\mathcal{B})$, cf. [Bos].

A stronger result has been proved by Bost [Bos]. As a consequence of the following theorem one finds that the conclusion of Theorem 8 still holds if only $G(\mathcal{B}) \cap \mathcal{A} = G(\mathcal{A})$ is satisfied.

Theorem 9 *If $\mathcal{A}$ and $\mathcal{B}$ are unital Banach algebras (or Fréchet algebras with open regular group and continuous inversion) and if $\varphi : \mathcal{A} \to \mathcal{B}$ is a continuous morphism with $\varphi(\mathcal{A})$ dense in $\mathcal{B}$ and $\varphi^{-1}(G(\mathcal{B})) = G(\mathcal{A})$ then the induced maps*

$$\varphi_n : M(n,\mathcal{A}) \ni (a_{ij}) \mapsto (\varphi(a_{ij})) \in M(n,\mathcal{B})$$

define homotopy equivalences

$$\varphi_n : Id(M(n,\mathcal{A})) \to Id(M(n,\mathcal{B}))$$

and

$$\varphi_n : GL(n,\mathcal{A}) \to GL(n,\mathcal{B})$$

for $n \geq 1$.

Bost has also considered many special cases including subalgebras of smooth elements with respect to a group action or crossed products by abelian groups.

4. The regular group of a C^*-algebra

In his book on Banach algebras [Ric] Rickart shows that Wintner's result about the connectivity of the unitary group carries over when the operator algebra $L(H_{\mathbb{C}})$ is replaced by a Rickart C^*-algebra. Indeed, the only condition to meet is that all maximal commutative C^*-subalgebras are of the form $C(X)$ with totally disconnected X. In particular, the unitary group of an AW^*-algebra and hence of a W^*-algebra are connected. If $\mathcal{M}$ is a real W^*-algebra without finite discrete part essentially the same arguments that were used by Putnam and Wintner show that its orthogonal group is also connected. This has been discovered by Ekman [Ekm] and Schreiber (unpublished) in 1974. Even Kuiper's theorem appears as a special case of a more general result about W^*-algebras. After a series of papers by Breuer, Singer, and Willgerodt dealing with countably decomposable ones, Brüning and Willgerodt [BW] finally proved contractibility for the regular group of any real or complex W^*-algebra of type I_∞, II_∞, or III.

Theorem 1 *If $\mathcal{M}$ is a W^*-algebra of infinite type then $G(\mathcal{M})$ and hence $U(\mathcal{M})$ are contractible.*

A finite continuous W^*-algebra behaves differently. Araki, L. Smith, and M.-S. Bae Smith [ASS] observed that the fundamental group $\pi_1(G\mathcal{M})$ of a type II_1-factor $\mathcal{M}$ is nontrivial, and Handelman proved that this extends to the nonfactorial case [Han1]. More precisely, they proved that

$$\pi_1(G(\mathcal{M})) \cong K_0(\mathcal{M}) \cong C(\Omega, \mathbb{C}),$$

if Ω is the Stonean space of the center of $\mathcal{M}$. Handelman also considered the fundamental group of the regular group of an $AF\,C^*$-algebra; the special case of UHF-algebras had been treated before by Singer, L. and M.-S. Bae Smith in unpublished work. In the factorial case (and also in case of a UHF-algebra) $K_0(\mathcal{M})$ is isomorphic to $\mathbb{R}$ (or to a dense subgroup of $\mathbb{R}$) and the isomorphism is induced by assigning the winding number $\tau(f) = \frac{1}{2\pi i}\int_{S^1} tr(f^{-1}df)$ to a differentiable loop f. Since the dimension group is generated by the traces of projections, taking simple loops of the form $t \mapsto e^{2\pi i p t}$, $0 \le t \le 1$, obviously shows that the induced map is surjective. The hard part is the proof of injectivity which proceeds by showing that any loop is homotopic to a concatenation of simple loops or even to a finite iteration of a single simple loop.

For a long time it was an open problem whether higher homotopy groups of the regular group $G(\mathcal{M})$ of a W^*-algebra $\mathcal{M}$ of type II_1 could serve in a finer classification of these algebras. However, this turned out not to be the case as shown by the author in 1983 [Srd1] for complex and in 1985 [Srd2] for real W^*-algebras.

Theorem 2 *If $\mathcal{M}$ is a complex W^*-algebra of type II_1 then*

$$\pi_k(G(\mathcal{M})) \cong \begin{cases} K_0(\mathcal{M}), & k \equiv 1 \pmod 2 \\ 0, & k \equiv 0 \pmod 2. \end{cases}$$

If $\mathcal{M}$ is a purely real W^*-algebra of type II_1 then

$$\pi_k(G(\mathcal{M})) \cong \begin{cases} K_0(\mathcal{M}), & k \equiv 3 \pmod 4 \\ 0, & k \equiv 0, 1, 2 \pmod 4. \end{cases}$$

The proof relies essentially on a stabilization theorem similar to Theorem 7 of section 3. It uses the fact that the regular group $G(\mathcal{M})$ of a type II_1 W^*-algebra $\mathcal{M}$ is dense in $\mathcal{M}$ which means that the (topological) stable rank is equal to 1, and that any projection p in $\mathcal{M}$ can be halved, i.e. $p = p_1 + p_2$ with two equivalent orthogonal projections p_1, p_2. The periodicity comes from the generalized Bott theorem.

From these results one easily deduces the homotopy type of homogeneous spaces relative to a W^*-algebra factor. Conjectured by Breuer and proved in [EFT] the Grassmannian of a continuous factor has simply connected components. The proof uses the fact mentioned before, that any loop in $U(\mathcal{M})$ is homotopic to the iterate of a simple loop. In [Srd3] we obtained the following conclusive results.

Theorem 3 *Let $\mathcal{M}$ be a continuous W^*-algebra factor, $Q = Q^* = Q^{-1} \in \mathcal{M}$, and, if $\mathcal{M}$ is purely real, $J^* = -J = J^{-1} \in \mathcal{M}$. By $U(\mathcal{M}, Q) = U(Q, H_{\mathbb{F}}) \cap \mathcal{M}$ we denote the group of Q-unitary elements relative to $\mathcal{M}$ and by $Sp(\mathcal{M}, J) = Sp(J, H_{\mathbb{R}}) \cap \mathcal{M}$ the symplectic group relative to $\mathcal{M}$. If, moreover, $p = 1/2(1+Q)$, $q = v^*v$, where $J = v - v^*$, then*

$$\pi_k(U(\mathcal{M}, Q)) \cong \pi_k(U(p\mathcal{M}p)) \oplus \pi_k(U((1-p)\mathcal{M}(1-p)))\,, \quad k \geq 0$$

and

$$\pi_k(Sp(\mathcal{M}, J)) \cong \pi_k(U(q\mathcal{M}q)_{\mathbb{C}})\,, \quad k \geq 0\,.$$

If $\mathcal{M}$ is a type II_1-factor, p a nontrivial projection then the component $Gr_p(\mathcal{M})$ of the Grassmannian that contains p has homotopy groups

$$\pi_k(Gr_p(\mathcal{M})) \cong \begin{cases} \mathbb{R}\,, & k \equiv 0 \pmod{2d} \\ 0\,, & k \text{ else}, \end{cases}$$

where $d = 1$ if $\mathcal{M}$ is complex, and $d = 2$ if $\mathcal{M}$ is purely real. The Banach homogeneous space $U_-(\mathcal{M})$ of skew-adjoint orthogonal elements in a purely real II_1-factor has homotopy groups

$$\pi_k(U_-(\mathcal{M})) \cong \begin{cases} \mathbb{R}\,, & k \equiv 2 \pmod 4 \\ 0\,, & k \text{ else.} \end{cases}$$

Before we come back to the homotopy type of the regular group in other special cases we want to take a look at K-theory. Most of the missing references can be found in [Bla]. K-theory has entered operator theory in the mid-seventies when Elliott used the dimension group ($= K_0(\mathcal{A})$) in order to classify $AF\,C^*$-algebras. For example, the UHF-algebras $\mathcal{A}_{(q)}$ defined as direct C^*-limits $(M(n_k, \mathbb{F}), i_k)$ of matrix algebras $M(n_k, \mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ where $n_{k+1} = q_k n_k$, $q_k \in \mathbb{N}$, and $i_k : M(n_k, \mathbb{F}) \ni x \mapsto x \otimes 1_{q_k} \in M(n_{k+1}, \mathbb{F})$, have K-groups

$$K_0(\mathcal{A}_{(q)}) \cong \mathbb{Z}_{(q)} = \lim \tfrac{1}{n_k}\mathbb{Z}, \quad K_1(\mathcal{A}_{(q)}) = 0,$$

if $\mathbb{F} = \mathbb{C}$, and

$$KO_i(\mathcal{A}_{(q)}) \cong \begin{cases} \mathbb{Z}_{(q)}, & i \equiv 0 \pmod 4, \\ \mathbb{Z}_2, & i \equiv 1, 2 \pmod 8,\ 2^\infty \not\mid (q), \\ 0, & i \text{ else}, \end{cases}$$

if $\mathbb{F} = \mathbb{R}$, using the fact [Han1] that the K-functor commutes with direct C^*-limits. Here (q) denotes the formal product $\prod q_k$, and $2^\infty \not| (q)$ means that 2 does occur at most finitely often in its formal prime decomposition. The $AF\, C^*$-algebras $\mathcal{A}^\theta$ defined as direct limits $(M(n_k, \mathbb{F}) \oplus M(m_k, \mathbb{F}), \phi_k)$, where n_k and m_k are defined recursively by $n_1 = m_1 = 1$, $n_{k+1} = a_k n_k + m_k$, $m_{k+1} = n_k$ using the expansion of the irrational number θ as a continued fraction, $\theta = [a_1; a_2, a_3, \ldots]$, and where $\phi_k(x, y) = (x \otimes 1_{a_k} \oplus y, x)$, have K-groups

$$K_0(\mathcal{A}^\theta) \cong \mathbb{Z} + \theta\mathbb{Z}, \quad K_1(\mathcal{A}^\theta) = 0,$$

if $\mathbb{F} = \mathbb{C}$, and

$$KO_i(\mathcal{A}^\theta) \cong \begin{cases} \mathbb{Z} + \theta\mathbb{Z}, & i \equiv 0 \pmod 4, \\ \mathbb{Z}_2^2, & i \equiv 1, 2 \pmod 8, \\ 0, & i \,\text{else}, \end{cases}$$

if $\mathbb{F} = \mathbb{R}$ [Srd4]. The classification of real $AF\, C^*$-algebras using real K-theory has been achieved by Giordano [Gio].

$AF\, C^*$-algebras are direct limits of zero-dimensional homogeneous algebras. Direct limits $\mathcal{B}_{(q)}$ of one-dimensional algebras $(M(n_k, C(S^1, \mathbb{F})), i_k)$ with i_k a p_k-times around imbedding have been considered by Bunce and Deddens, cf. [Bla]. Analogously to UHF-algebras one gets

$$K_0(\mathcal{B}_{(q)}) \cong \mathbb{Z}_{(q)}, \quad K_1(\mathcal{B}_{(q)}) \cong \mathbb{Z},$$

if $\mathbb{F} = \mathbb{C}$, and

$$KO_i(\mathcal{B}_{(q)}) \cong \begin{cases} \mathbb{Z}_{(q)}(+\mathbb{Z}_2), & i \equiv 0 \pmod 8 \,(\text{if } 2^\infty \not| (q)), \\ \mathbb{Z}_{(q)}, & i \equiv 4 \pmod 8, \\ \mathbb{Z}_2(+\mathbb{Z}_2), & i \equiv 1 \pmod 8 \,(\text{if } 2^\infty \not| (q)), \\ \mathbb{Z}_2, & i \equiv 2 \pmod 8, \text{if } 2^\infty \not| (q), \\ \mathbb{Z}, & i \equiv 3 \pmod 4, \\ 0, & i \,\text{else}, \end{cases}$$

if $\mathbb{F} = \mathbb{R}$ [Srd4]. For a large class of complex direct C^*-limits of one-dimensional algebras (those of real rank zero) Elliot has recently given also a classification using K-theory [Ell].

The first highlight in operator K-theory was obtained by Rieffel, Pimsner, and Voiculescu who used the dimension group to distinguish (up to Morita equivalence) between the irrational rotation algebras $\mathcal{A}_\theta$ for various θ, and, in particular, to exhibit nontrivial projections in $\mathcal{A}_\theta$ if $\theta \notin \mathbb{Q}$. The crucial device invented by Pimsner and Voiculescu [PV1] for this purpose was the six-term exact sequence for crossed products by $\mathbb{Z}$.

Theorem 4 *If $\mathcal{A}$ is a C^*-algebra provided with a $\mathbb{Z}$-action $\alpha : \mathbb{Z} \to Aut(\mathcal{A})$ and $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ is the corresponding crossed product then one has the cyclic exact sequence*

$$\begin{array}{ccccc} K_0(\mathcal{A}) & \stackrel{1-\alpha_*}{\to} & K_0(\mathcal{A}) & \stackrel{i_*}{\to} & K_0(\mathcal{A} \times_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{A} \rtimes_\alpha \mathbb{Z}) & \stackrel{i_*}{\leftarrow} & K_1(\mathcal{A}) & \stackrel{1-\alpha_*}{\leftarrow} & K_1(\mathcal{A}). \end{array}$$

Applying the Pimsner-Voiculescu sequence to the crossed product $\mathcal{A}_\theta = C(S^1,\mathbb{C})\rtimes_\alpha\mathbb{Z}$, where $\mathbb{Z}$ acts on $C(S^1,\mathbb{C})$ by rotating the argument of a function by $e^{2\pi i\theta}$ gives the following result.

Corollary 1 *If $\theta \notin \mathbb{Q}$ and if $\mathcal{A}_\theta = C(S^1,\mathbb{C})\rtimes_\alpha\mathbb{Z}$ is the corresponding irrational rotation algebra then*

$$K_0(\mathcal{A}_\theta) \cong \mathbb{Z} + \theta\mathbb{Z} \quad \textit{and} \quad K_1(\mathcal{A}_\theta) \cong \mathbb{Z}^2.$$

An analogous cyclic exact sequence with 24 terms holds in the real case [Ros3], [Srd4], and [Sty] and gives:

Corollary 2 *If $\theta \notin \mathbb{Q}$ and if $\mathcal{A}_\theta^\mathbb{R} = C(S^1,\mathbb{R})\rtimes_\alpha\mathbb{Z}$ is the corresponding real irrational rotation algebra then*

$$KO_i(\mathcal{A}_\theta^\mathbb{R}) \cong \begin{cases} \mathbb{Z}+\theta\mathbb{Z}+\mathbb{Z}_2, & i \equiv 0 \pmod 8, \\ \mathbb{Z}+\mathbb{Z}_2^3, & i \equiv 1 \pmod 8, \\ \mathbb{Z}_2^3, & i \equiv 2 \pmod 8, \\ \mathbb{Z}+\mathbb{Z}_2, & i \equiv 3 \pmod 8, \\ \mathbb{Z}+\theta\mathbb{Z}, & i \equiv 4 \pmod 8, \\ \mathbb{Z}, & i \equiv 5,7 \pmod 8, \\ 0, & i \equiv 6 \pmod 8. \end{cases}$$

To obtain these corollaries one has to notice that the maps α_* induced by the action are the identity (because the phase $e^{2\pi i\theta}$ can be deformed to 1 and the K-groups are not affected by the deformation [AP]) and to prove that the so obtained short exact sequences split. Note that the isomorphism of K_0 with a subgroup of $\mathbb{R}$ is induced by the unique tracial state of these algebras; cf. [PV1], [Pim1]. Crossed products coming from torus actions are treated in [Rou].

Just as $\mathcal{A}_\theta$ can be considered as a deformation of the commutative C^*-algebra $C(T^2)$ one can deform the relations that determine, e.g., $S^3 = SU(2)$ as a submanifold of $\mathbb{C}^2$ by inserting a phase factor. Thus, for $\theta \notin \mathbb{Q}$ the noncommutative 3-sphere S_θ^3 can either be defined by suitable generators and relations or more geometrically by gluing two noncommutative solid tori [Mat]: For $D = \{z \in \mathbb{C} \mid |z| \le 1\}$ let $\mathcal{D}_\theta = C(D)\rtimes_\alpha\mathbb{Z}$ with α as in Corollary 1. Then one has a canonical surjection $\pi_\theta : \mathcal{D}_\theta \to \mathcal{A}_\theta$. Define an isomorphism $\rho : \mathcal{A}_{-\theta} \to \mathcal{A}_\theta$ by $\rho(\hat{u}) = v$ and $\rho(\hat{v}) = u$ on the corresponding generators and let

$$S_\theta^3 = \{(a,b) \in \mathcal{D}_{-\theta} \oplus \mathcal{D}_\theta \mid \rho(\pi_{-\theta}(a)) = \pi_\theta(b)\}.$$

The computation of the corresponding K-groups uses the following Mayer-Vietoris sequence for a pull-back

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{g_1} & \mathcal{B}_1 \\ g_2 \downarrow & & \downarrow f_1 \\ \mathcal{B}_2 & \xrightarrow{f_2} & \mathcal{C} \end{array}$$

of C^*-algebras where $\mathcal{D} = \{(b_1,b_2) \in \mathcal{B}_1 \oplus \mathcal{B}_2 \mid f_1(b_1) = f_2(b_2)\}$, f_1, f_2 are surjective, and where $g_i : \mathcal{D} \to \mathcal{B}_i$, $i = 1,2$, are the restrictions of the canonical projections, cf. [Bla] and [Srd4] (also for more applications).

Theorem 5 *Any pull-back diagram of C^*-algebras gives rise to a long exact sequence*

$$\to K_n(\mathcal{D}) \overset{(g_{1*},g_{2*})}{\longrightarrow} K_n(\mathcal{B}_1) \oplus K_n(\mathcal{B}_2) \overset{f_{2*}-f_{1*}}{\longrightarrow} K_n(\mathcal{C}) \overset{\alpha}{\longrightarrow} K_{n-1}(\mathcal{D}) \to .$$

Corollary 1 *If $\theta \notin \mathbb{Q}$ then the K-groups of S^3_θ are*

$$K_0(S^3_\theta) \cong \mathbb{Z} \cong K_1(S^3_\theta).$$

The real case is more involved. Here one has to "glue" real subalgebras of irrational rotation algebras with different real structure. The real structure is defined on the generators by $\bar{u} = u^*$, $\bar{v} = v$ in case of $\mathcal{A}_\theta$ and by $\bar{\hat{v}} = \hat{v}^*$ and $\bar{\hat{u}} = \hat{u}$ in case of $\mathcal{A}_{-\theta}$. Now, since the latter C^*-algebra is not a crossed-product, the corresponding real form for $\mathcal{D}_{-\theta}$ is the "cone" $\{f \in C([0,1], \mathcal{A}^{\mathbb{R}}_{-\theta} \mid f(0) \in C(S^1, \mathbb{R})\}$. Then one obtains for the real noncommutative 3-sphere $S^{3\mathbb{R}}_\theta$:

Corollary 2 *If $\theta \notin \mathbb{Q}$ then the K-groups of $S^{3\mathbb{R}}_\theta$ are*

$$K_0(S^{3\mathbb{R}}_\theta) \cong \begin{cases} \mathbb{Z} + \mathbb{Z}_2, & i \equiv 0 \pmod 8, \\ \mathbb{Z}_2^2, & i \equiv 1 \pmod 8, \\ \mathbb{Z}_2, & i \equiv 2 \pmod 8, \\ \mathbb{Z}, & i \equiv 3, 4, \text{ or } 7 \pmod 8, \\ 0, & i \text{ else}. \end{cases}$$

More generally, one can define noncommutative lens spaces $L_\theta(p,q)$ using the gluing isomorphism that is given by

$$\rho_A(u) = e^{-k\ell\pi i\theta} v^k u^\ell = \hat{v} \quad \text{and} \quad \rho_A(v) = e^{-pq\pi i\theta} v^q u^p = \hat{u}$$

with $A = \begin{pmatrix} q & k \\ p & \ell \end{pmatrix} \in SL(2, \mathbb{Z})$. Slightly more elaborate calculations yield

$$K_0(L_\theta(p,q)) \cong \mathbb{Z}_p + \mathbb{Z}, \quad K_1(L_\theta(p,q) \cong \mathbb{Z},$$

$$K_0(L_\theta(0,1)) \cong \mathbb{Z}^2, \quad K_1(L_\theta(p,q) \cong \mathbb{Z}^2,$$

in the complex case (cf. [MT]), and

$$KO_i(L^{\mathbb{R}}_\theta(p,q)) \cong \begin{cases} \mathbb{Z} + \mathbb{Z}_p + \mathbb{Z}_2, & i \equiv 0 \pmod 8 \\ \mathbb{Z}_2^2 + \mathbb{Z}_{(p,2)}, & i \equiv 1 \pmod 8, \\ \mathbb{Z}_2 + \mathbb{Z}^2_{(p,2)}, & i \equiv 2 \pmod 8, \\ \mathbb{Z} + \mathbb{Z}_{(p,2)}, & i \equiv 3 \pmod 8, \\ \mathbb{Z} + \mathbb{Z}_p, & i \equiv 4 \pmod 8, \\ \mathbb{Z}, & i \equiv 7 \pmod 8, \\ 0, & i \,\text{else}, \end{cases}$$

in the real case [Sdr5].

As a final application of the Mayer-Vietoris sequence we give the K-groups of C^*-algebras associated with compact orbifolds of dimension 2, i.e., singular algebraic curves. Topologically such a curve is an oriented compact surface of genus g whose singular locus consist of a finite set of points of multiplicity n_j, i.e. at such a point n_j branches meet. The complex case has been treated by Carla Farsi [Far] the real case has been determined by the author [Srd5]. Denoting by ℓ the number of singular points, and $\sigma = 1 + \sum_{j=1}^{\ell}(n_j - 1)$ one obtains

Corollary 3 *For an orbifold $T_{g,\ell}$ of genus g with ℓ singular points of multiplicity n_j the K-groups of the corresponding C^*-algebras $C^*(T_{g,\ell})$ and $C^*(T^{\mathbb{R}}_{g,\ell})$ are given by*

$$K_0(C^*(T_{g,\ell})) \cong \mathbb{Z}^{1+\sigma} \quad \textit{and} \quad K_1(C^*(T_{g,\ell})) \cong \mathbb{Z}^{2g}$$

and by

$$KO_i(C^*(T^{\mathbb{R}}_{g,\ell})) \cong \begin{cases} \mathbb{Z}^\sigma + \mathbb{Z}_2^{2g+1}, & i \equiv 0 \pmod 8 \\ \mathbb{Z}_2^{2g+\sigma}, & i \equiv 1 \pmod 8, \\ \mathbb{Z} + \mathbb{Z}_2^\sigma, & i \equiv 2 \pmod 8, \\ \mathbb{Z}^{2g}, & i \equiv 3 \pmod 8, \\ \mathbb{Z}^\sigma, & i \equiv 4 \pmod 8, \\ 0, & i \equiv 5 \pmod 8, \\ \mathbb{Z}, & i \equiv 6 \pmod 8, \\ \mathbb{Z}^{2g} + \mathbb{Z}_2, & i \equiv 7 \pmod 8, \end{cases}$$

Another class of examples is obtained by extensions, i.e. C^*-algebras $\mathcal{A}$ that fit into a short exact sequence

$$0 \to \mathcal{J} \to \mathcal{A} \to \mathcal{B} \to 0.$$

From this sequence one obtains a long exact sequence of K-groups (of period 6 in the complex and of period 24 in the real case)

$$\to K_n(\mathcal{J}) \to K_n(\mathcal{A}) \to K_n(\mathcal{B}) \to K_{n-1}(\mathcal{J}) \to \ .$$

As an application we consider the Toeplitz C^*-algebra $\mathcal{T}(S^{2n-1})$ and the Calderon-Zygmund algebra $CZ(S^k)$ of zero-order pseudodifferential operators on S^k which are both extensions of the compacts by a commutative C^*-algebra, viz.

$$0 \to \mathcal{K} \to \mathcal{T}(S^{2n-1}) \to C(S^{2n-1}) \to 0,$$

and

$$0 \to \mathcal{K} \to CZ(S^k) \to C(S^*S^k) \to 0,$$

where S^*S^k is the cosphere bundle over S^k. Now $K_n(C(S^k)) = K_n(\mathbb{C}) + K_{n+k}(\mathbb{C})$ and

$$K_0(C(S^*S^k)) \cong \begin{cases} \mathbb{Z} + \mathbb{Z}_2, & k \text{ even} \\ \mathbb{Z} + \mathbb{Z}, & k \text{ odd} \end{cases} \quad \text{and} \quad K_1(C(S^*S^k)) \cong \begin{cases} \mathbb{Z}, & k \text{ even} \\ \mathbb{Z} + \mathbb{Z}, & k \text{ odd}, \end{cases}$$

so that

$$K_0(\mathcal{T}(S^{2n-1})) \cong \mathbb{Z} \cong K_1(\mathcal{T}(S^{2n-1}))$$

and [Les]

$$K_0(CZ(S^{2n})) \cong \mathbb{Z} + \mathbb{Z}_2, \quad K_1(CZ(S^{2n})) = 0,$$
$$K_0(CZ(S^{2n-1})) \cong \mathbb{Z}^2, \quad K_1(CZ(S^{2n-1})) \cong \mathbb{Z}.$$

These groups can in turn be used to compute the K-groups of other Toeplitz C^*-algebras. For the C^*-algebra $\mathcal{T}(S^{2n_1-1} \times \cdots \times S^{2n_k-1})$ of Toeplitz operators on the Hardy space $H^2(S^{2n_1-1} \times \cdots \times S^{2n_k-1})$ with continuous symbols or $\mathcal{T}(L_n)$ the corresponding algebra on the Hardy space over the Lie spheres L_n the K-group $K_0(\mathcal{T})$ is isomorphic to $\mathbb{Z}$ and $K_1(\mathcal{T})$ is trivial [Deu1], [Les]. In the second case one has again an extension

$$0 \to \mathcal{K}(H^2(S^1)) \otimes CZ(S^{n-1}) \to \mathcal{T}(L_n) \to C(L_n) \to 0$$

and uses

$$K_0(C(L_k)) \cong \begin{cases} \mathbb{Z} + \mathbb{Z}, & k \,\text{even} \\ \mathbb{Z}, & k \,\text{odd} \end{cases} \quad \text{and} \quad K_1(C(L_k)) \cong \begin{cases} \mathbb{Z} + \mathbb{Z}, & k \,\text{even} \\ \mathbb{Z} + \mathbb{Z}_2, & k \,\text{odd}, \end{cases}$$

while the first is a direct consequence of the Künneth short split exact sequence

$$0 \to K_*(\mathcal{A}) \otimes K_*(\mathcal{B}) \to K_*(\mathcal{A} \otimes \mathcal{B}) \to Tor_1^{\mathbb{Z}}(K_*(\mathcal{A}), K_*(\mathcal{B})) \to 0.$$

The proof given in [Srd4] and in [Sty] for the real Pimsner-Voiculescu sequence and given by Joachim Cuntz in [Cun2,5] as an alternative proof in the complex case is based on the homotopy properties of the regular group of another class of C^*-algebras, the so-called Cuntz algebras $\mathcal{O}_n$, $n \geq 2$, first defined by Cuntz in 1977.

Theorem 6 *Let $\mathcal{O}_n^{(\mathbb{R})}$ be the (real) C^*-algebra generated by n (real) partial isometries S_j with $S_j^* S_j = 1$ and $\sum_{j=1}^{n} S_j S_j^* = 1$, then*

$$K_i(\mathcal{O}_n) \cong \begin{cases} \mathbb{Z}_{n-1}\,, & i \equiv 0 \pmod 2, \\ 0\,, & i \equiv 1 \pmod 2, \end{cases}$$

and

$$KO_i(\mathcal{O}_n^{\mathbb{R}}) \cong \begin{cases} \mathbb{Z}_{n-1}\,, & i \equiv 0 \pmod 4 \\ \mathbb{Z}_2\,, & i \equiv 1,3, \pmod 8, n \textit{ odd}, \\ \mathbb{Z}_2^2\,, & i \equiv 2 \pmod 8, n \textit{ odd}, \\ 0\,, & i \textit{ else} \end{cases}$$

We refer to [Cun3] and [Srd4] for the K-groups of the more general Cuntz-Krieger C^*-algebras.

A generalization of Bott's periodicity theorem has been obtained by Connes for actions $\alpha : \mathbb{R} \to Aut(\mathcal{A})$ [Con] - see [Ksp2] and [Srd4] for the proof of the real version.

Theorem 7 *For any complex C^*-algebra $\mathcal{A}$ and any $\mathbb{R}$-action α on $\mathcal{A}$ one has isomorphisms*

$$K_i(\mathcal{A} \rtimes_\alpha \mathbb{R}) \cong K_{1-i}(\mathcal{A}), \quad i = 0, 1 .$$

If $\mathcal{A}$ is a real C^*-algebra there are isomorphisms

$$KO_i(\mathcal{A} \rtimes_\alpha \mathbb{R}) \cong KO_{i-1}(\mathcal{A}), \quad i \in \mathbb{Z} .$$

His main purpose was to compute K-groups of group C^*-algebras $C^*(G)$ and $C^*_{red}(G)$ where G is a connected Lie group and $C^*(G)$ denotes the completion of the convolution algebra $L^1(G)$ with respect to the greatest C^*-norm and $C^*_{red}(G)$ with respect to the norm induced by left regular representation on $L^2(G)$. The real group C^*-algebras $C^*(G, \mathbb{R})$ and $C^*_{red}(G, \mathbb{R})$ are the closure of $L^1(G, \mathbb{R})$ under the respective norm in $C^*(G)$ and $C^*_{red}(G)$. For a compact group G one has

$$K_0(C^*(G)) \cong \mathbb{Z}\hat{G} = R(G), \quad K_1(C^*(G)) = 0,$$

$R(G)$ the representation ring of G, and, more generally, for an action α of G on a C^*-algebra $\mathcal{A}$ the so-called Green-Julg theorem holds:

$$K_i(\mathcal{A} \rtimes_\alpha G) \cong K_i^G(\mathcal{A}), \quad i = 0, 1.$$

For G a noncompact connected Lie group with maximal compact subgroup H Connes conjectured isomorphisms [BaC]

$$K_i(C^*_{red}(G)) \to K_H^{i + \dim G/H}(\{pt\}), \quad i = 0, 1.$$

This conjecture has since been confirmed in several special cases. By successive application of Theorem 7 one gets [Con] for a connected simply connected solvable Lie group G that

$$K_i(C^*_{red}(G)) \cong K^{i+\dim G}(\{pt\}) \cong \begin{cases} \mathbb{Z}, & i \equiv \dim G \pmod 2, \\ 0, & i \not\equiv \dim G \pmod 2, \end{cases}$$

or more generally for a G-C^*-algebra $\mathcal{A}$

$$K_i(\mathcal{A} \rtimes G) \cong K_{i+\dim G}(\mathcal{A}).$$

The conjecture is also true if G is nilpotent, or a motion group, i.e. $G = V \rtimes K$, V a vector group and K a connected compact Lie group acting linearly on V, or if G is semi-simple of rank 1 [Ros1,2].
If G is a connected reductive linear group and H a maximal compact subgroup then

$$K_i(C^*_{red}(G)) \cong \begin{cases} R_{spin}(H), & i \equiv \dim(G/H) \pmod 2, \\ 0, & i \text{ else}, \end{cases}$$

where $R_{spin}(H)$ is the $R(H)$-module of "spinoral" representations of H. This has been proved in special cases by Penington and Plymen and by Valette, and in full generality by Wasserman [Was].

Another important class of C^*-algebras are the group C^*-algebras of discrete groups. Again a great impact has been made by Pimsner and Voiculescu who considered crossed products by actions of finitely generated free groups. There is a series of papers devoted to the K-theory of such group C^*-algebras and of their crossed products by Anderson and Paschke, Cuntz, Lance, Kasparov, Natsume, and finally by Pimsner [Pim2]. Pimsner's paper contains all the previous ones as special cases. The outcome is a kind of Mayer-Vietoris sequence for the K-groups of the group C^*-algebra of a group acting without involution on a tree and whose building blocks are the stabilizers of edges and vertices of the tree. We do not state the exact sequence here but emphasize that it holds in the real and in the complex case, [Srd5]. Using this exact sequence the K-groups of many group C^*-algebras can be computed.

$K_i(C^*_{red}(\Gamma))$ Γ i	F_k	$\mathbb{Z}_k *_{\mathbb{Z}_\ell} \mathbb{Z}_m$	Γ_g	Σ_k	H_d	$\mathbb{Q} * \mathbb{Q}$
0	$\mathbb{Z}$	$\mathbb{Z}^{k+m-\ell}$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}$
1	$\mathbb{Z}^k$	0	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{k-1}+\mathbb{Z}_2$	$\mathbb{Z}^3$	$\mathbb{Q}^2$
$KO_i(C^*_{red}(\Gamma))$ i						
0	$\mathbb{Z}$	$\mathbb{Z}^{k+m-\ell}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}^k+\mathbb{Z}_2$	$\mathbb{Z}_2^{k+m-\ell}$	$\mathbb{Z}^{2g}+\mathbb{Z}_2$	$\mathbb{Z}^{k-1}+\mathbb{Z}_2^2$	$\mathbb{Z}^2+\mathbb{Z}_2$	$\mathbb{Q}^2+\mathbb{Z}_2$
2	$\mathbb{Z}_2^{k+1}$	$\mathbb{Z}_2^{k+m-\ell}$	$\mathbb{Z}+\mathbb{Z}_2^{2g+1}$	$\mathbb{Z}_2^{k+1}$	$\mathbb{Z}^2+\mathbb{Z}_2^3$	$\mathbb{Z}_2$
3	$\mathbb{Z}_2^k$	0	$\mathbb{Z}_2^{2g+1}$	$\mathbb{Z}_2^{k+1}$	$\mathbb{Z}+\mathbb{Z}_2^4$	0
4	$\mathbb{Z}$	$\mathbb{Z}^{k+m-\ell}$	$\mathbb{Z}+\mathbb{Z}_2$	$\mathbb{Z}+\mathbb{Z}_2$	$\mathbb{Z}+\mathbb{Z}_2^3$	$\mathbb{Z}$
5	$\mathbb{Z}^k$	0	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{k-1}+\mathbb{Z}_2$	$\mathbb{Z}^2+\mathbb{Z}_2$	$\mathbb{Q}^2$
6	0	0	$\mathbb{Z}$	0	$\mathbb{Z}^2$	0
7	0	0	0	0	$\mathbb{Z}$	0

Here F_k denotes the free group on k generators, Γ_g and Σ_k are the fundamental groups of oriented resp. nonorientable compact surfaces, and H_d is the discrete Heisenberg group defined as before but with $x, y, z \in \mathbb{Z}$. We have only listed K-groups of the reduced C^*-algebras. But in all cited examples they coincide with the K-groups of the corresponding full C^*-algebras, because the groups are K-amenable, a notion introduced by Cuntz for discrete groups and by Julg and Valette for locally compact continuous groups. Using different methods Kasparov and Skandalis [KSk] obtain a still more general result for groups operating on a Bruhat-Tits building.

We also like to mention that K-theory helped to confirm an old standing conjecture by Kadison on the existence of nontrivial idempotents in the group C^*-algebra of a torsion-free discrete group in many special cases. This conjecture is related to another one: If Γ is torsion-free, then the canonical trace τ induces a surjection $\tau_* : K_0(C^*_{red}(\Gamma)) \to \mathbb{Z}$. While this map is in general not one-to-one, e.g. for $\Gamma = \Gamma_g$, our calculations indicate the following strengthening: If Γ is torsion-free, then the canonical trace τ restricted to the real group C^*-algebra induces an isomorphism $\tau_* : KO_0(C^*_{red}(\Gamma, \mathbb{R})) \to \mathbb{Z}$.

Now we come back to nonstable K-theory, i.e. the homotopy groups of the regular group of a C^*-algebra $\mathcal{A}$ itself. In the previous section we have already noted a connection between the size of the stable range

$$SR(\mathcal{A}) = \{(n,k) \in \mathbb{Z}_+ \times \mathbb{N} | \, k \geq k(n)\},$$

with $k(n) = \min\{k|\pi_n(GL(k,\mathcal{A})) \cong \pi_n(GL(\infty,\mathcal{A}))\}$, and the stable rank $sr(\mathcal{A})$ if this is finite. Elaborating on the proof of Theorem 2, Rieffel found the following general results [Rie2].

Theorem 8 *If $\mathcal{A}$ is a unital C^*-algebra, $p \in \mathbb{N}$, and $csr(C(T^k,\mathcal{A})) \leq p$ for all $k \geq 0$, then*

$$\pi_k(GL(n,\mathcal{A})) \cong \begin{cases} K_1(\mathcal{A}), & k \text{ even}, \\ K_0(\mathcal{A}), & k \text{ odd}, \end{cases}$$

for all $n \geq p-1$ and $k \geq 0$.

Here $csr(\mathcal{A})$ denotes the connected stable rank, i.e. the least integer m such that $GL^0(n,\mathcal{A})$ acts transitively on

$$Lg(n,\mathcal{A}) = \{(a_1,\ldots,a_n) \in \mathcal{A}^n | \sum b_i a_i = 1 \text{ for some } (b_1,\ldots,b_n) \in \mathcal{A}^n\}$$

for all $n \geq m$.
As corollaries he obtains that $\mathcal{A}$ has full stable range if $\mathcal{A}$ is tsr-boundedly divisible, i.e. if there is a constant c, such that for every m there is an $n \geq m$ and a C^*-algebra $\mathcal{B}$ with $\mathcal{A} \cong M(n,\mathcal{B})$ and $tsr(\mathcal{B}) \leq c$. Examples of trs-boundedly divisible C^*-algebras are type II_1 AW^*-algebras and tensor products $\mathcal{A}\otimes\mathcal{B}$ where $\mathcal{B}$ is a unital divisible AF C^*-algebra (e.g. a UHF-algebra) and $\mathcal{A}$ is unital with $tsr(\mathcal{A}) < \infty$ or $\mathcal{A} = C(X) \rtimes_\alpha \Gamma$ is a crossed product with a discrete solvable group Γ acting on a compact space X. Thomsen has shown [Tho] that $\mathcal{A} \otimes \mathcal{B}$ has also full stable range if $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{B}$ is an AF C^*-algebra with $K_0(\mathcal{B})$ having large denominators in the sense of [Nis], in particular, if $\mathcal{B}$ is infinite dimensional and simple.
Moreover, he considers the groups of quasi-invertibles and of quasi-unitaries, $G_q(\mathcal{A})$ resp. $U_q(\mathcal{A})$, in a nonunital complex C^*-algebra $\mathcal{A}$, and shows that an exact sequence

$$0 \to \mathcal{J} \xrightarrow{i} \mathcal{A} \xrightarrow{p} \mathcal{B} \to 0$$

of C^*-algebras gives rise to a long exact sequence of homotopy groups

$$\to \pi_k(U_q(\mathcal{J})) \xrightarrow{i_*} \pi_k(U_q(\mathcal{A})) \xrightarrow{p_*} \pi_k(U_q(\mathcal{B})) \to \pi_{k-1}(U_q(\mathcal{J})) \to \cdots \to \pi_0(U_q(\mathcal{B})).$$

Note that an analogous exact sequence can be obtained for the groups of quasi-invertibles. Then the proof uses the selection theorem of Michael at a crucial point, and leads to the same conclusion for an exact sequence of real C^*-algebras. If X is a compact space Thomsen obtains the following partial result ($C(X) = C(X,\mathbb{C})$).

Theorem 9 *If X is a compact Hausdorff space then*

$$\pi_n(U(C(X))) \cong H^{1-n}(X,\mathbb{Z}), \quad n \geq 0.$$

In particular, $\pi_1(U(C(X))) \cong \mathbb{Z}$, if X is connected, and $\pi_n(U(C(X))) = 0$ for $n \geq 2$.

As an immediate corollary we obtain the homotopy groups of the regular group of the Toeplitz C^*-algebra $\mathcal{T}(S^{2n-1})$ and of the Calderon-Zygmund algebra $CZ(S^n)$.

Corollary *For $n \geq 1$ and $\mathcal{A} = \mathcal{T}(S^{2n-1})$ or $CZ(S^{n+1})$ one has*

$$\pi_k(G(\mathcal{A})) = \begin{cases} \mathbb{Z}, & k \equiv 1 \pmod 2, k \neq 1, \\ 0, & k \equiv 0 \pmod 2, \\ \mathbb{Z}^2, & k = 1, \end{cases}$$

while

$$\pi_k(G(CZ(S^1))) = \begin{cases} \mathbb{Z}, & k \equiv 1 \pmod 2, k \neq 1 \textit{ or } k = 0, \\ 0, & k \equiv 0 \pmod 2, k \neq 0, \\ \mathbb{Z}^3, & k = 1. \end{cases}$$

The results for $CZ(S^1)$ can also be found in the papers by Khimshiashvili [Khi1,2]. Now in both cases the stable rank is equal to $n-1$ or to n, and there are homotopy groups of the regular group of matrix algebras which have torsion. To determine these homotopy groups is as hard as the computation of the homotopy groups of classical groups in the instable range.

Now the relation between stable rank and size of the stable range of a C^*-algebra is not completely settled. There are many examples of extremely noncommutative C^*-algebras with infinite stable rank and full stable range, e.g. purely infinite simple C^*-algebras (such as the Cuntz algebras $\mathcal{O}_A$) [Zha1-4] or $\mathcal{A} \otimes \mathcal{B}$, where $\mathcal{B}$ is an arbitrary C^*-algebra (in the nonunital case consider quasi-invertibles), and $\mathcal{A}$ is either $\mathcal{O}_n^{(\mathbb{R})}$ [Tho] ([Srd4]), a W^*-algebra $\mathcal{M}$ of infinite type, the multiplier algebra $M(\mathcal{D} \otimes \mathcal{K})$ of $\mathcal{D} \otimes \mathcal{K}$, where $\mathcal{D}$ is a C^*-algebra with a strictly positive element, or the corresponding Calkin algebra [Tho].

So it finally appears that the crucial property for full stable range is some kind of divisibility which is guaranteed if the algebra has a lot of projections. And if there are enough projections not equivalent to the identity, the regular group tends to be contractible, cf. section 2. The most important example here is the algebra $M(\mathcal{A} \otimes \mathcal{K}) \otimes \mathcal{B}$ where $\mathcal{A}$ has a strictly positive element and $\mathcal{B}$ is arbitrary. This generalization of Kuiper's theorem (which is $\mathcal{A} = \mathbb{C} = \mathcal{B}$) has been proved by Mingo, Kasimov, and Troitskii in special cases, by Cuntz and Higson [CH] when $\mathcal{B} = \mathbb{C}$, and by Thomsen [Tho] and Troitskii [Tro] in general (see [Srd4] for real C^*-algebras).

As in the case $\mathcal{A} = \mathbb{F}$ the general Kuiper theorem has applications to classifying spaces using generalized Fredholm operators, see [Min] and [Srd4]. Fredholm operators relative to the ideal of "compact" operators in a semifinite W^*-algebra (or AW^*-algebra) have been considered by [Bre], [Gew], [Php], [Rae], and [Srb], see also [CPh].

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